Flows of Spin(7)-structures

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Abstract

We consider flows of Spin(7)-structures. We use local coordinates to describe the torsion tensor of a Spin(7)-structure and derive the evolution equations for a general flow of a Spin(7)-structure Φ on an 8-manifold M. Specifically, we compute the evolution of the metric and the torsion tensor. We also give an explicit description of the decomposition of the space of forms on a manifold with Spin(7)-structure, and derive an analogue of the second Bianchi identity in Spin(7)-geometry. This identity yields an explicit formula for the Ricci tensor and part of the Riemann curvature tensor in terms of the torsion.

1 Introduction

This paper discusses general flows of Spin(7)-structures in a manner similar to the author's analogous results for flows of G_2-structures, which were studied in [8]. Many of the calculations are similar in spirit, although more involved, so we often omit proofs. The reader is advised to familiarize themselves with [8] first.

A general evolution of a Spin(7)-structure is described by a symmetric tensor h and a skew-symmetric tensor X satisfying some further algebraic condition, and it is only h which affects the evolution of the associated Riemannian metric. However, the evolution of the torsion tensor is determined by both h and X.

In Section 2, we review Spin(7)-structures, the decomposition of the space of forms, and the torsion tensor of a Spin(7)-structure. In Section 3 we compute the evolution equations for the metric and the torsion tensor for a general flow of Spin(7)-structures. In Section 4, we apply our evolution equations to derive a Bianchi-type identity in Spin(7)-geometry. This leads to an explicit formula for the Ricci tensor of a general Spin(7)-structure in terms of the torsion. An Appendix collects various identities in Spin(7)-geometry.

The notation used in this paper is identical to that of [8]. Throughout this paper, M is a (not necessarily compact) smooth manifold of dimension 8 which admits a Spin(7)-structure.

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2 Manifolds with Spin(7)-structure

In this section we review the concept of a Spin(7)-structure on a manifold M and the associated decompositions of the space of forms. More details about Spin(7)-structures can be found, for example, in [1, 4, 5, 6]. We also describe explicitly the torsion tensor associated to a Spin(7)-structure.

Consider an 8-manifold M with a Spin(7) structure Φ. The existence of such a structure is a *topological condition*. The space of 4-forms Φ on M which determine a Spin(7)-structure is a subbundle $\mathcal{A}$ of the bundle Ω^4 of 4-forms on M, called the bundle of *admissible* 4-forms. This is *not* a vector subbundle, and unlike the G_2 case, it is not even an open subbundle.

A Spin(7)-structure Φ determines a Riemannian metric g_Φ and an orientation in a non-linear fashion which we now describe. Let v be a non-zero tangent vector at a point p, and extend to $(v, e_1, \ldots, e_7)$ a local frame near p. We define

$$\begin{aligned} B_{ij}(v) &= ((e_i \lrcorner v \lrcorner \Phi) \wedge (e_j \lrcorner v \lrcorner \Phi) \wedge (v \lrcorner \Phi))\,(e_1, \ldots, e_7) \\ A(v) &= ((v \lrcorner \Phi) \wedge \Phi)\,(e_1, \ldots, e_7) \end{aligned}$$

Then the metric g_Φ is defined by

$$(g_\Phi(v,v))^2 = -\frac{7^3}{6^{\frac{7}{3}}}\,\frac{(\det B_{ij}(v))^{\frac{1}{3}}}{A(v)^3} \tag{2.1}$$

More details can be found in [6], although that paper uses a different orientation convention (see also [7].) We will not have need for this explicit formula.

The metric g_Φ and orientation (determined by the volume form) determine a Hodge star operator $*$, and the 4-form Φ is *self-dual*. That is, $*\Phi = \Phi$. The metric also determines the Levi-Civita connection ∇, and the manifold (M, Φ) is called a Spin(7) manifold if $\nabla\Phi = 0$. This is a non-linear partial differential equation for Φ, since ∇ depends on g which depends non-linearly on Φ. Such manifolds (where Φ is parallel) have Riemannian holonomy $\mathrm{Hol}_g(M)$ contained in the group $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$. A parallel Spin(7)-structure is also called *torsion-free*.

2.1 Decomposition of the space of forms

The existence of a Spin(7)-structure Φ on M (with no condition on $\nabla\Phi$) determines a decomposition of the spaces of differential forms on M into irreducible Spin(7) representations. We will see explicitly that the spaces Ω^2, Ω^3, and Ω^4 decompose as

$$\Omega^2 = \Omega^2_7 \oplus \Omega^2_{21} \qquad \Omega^3 = \Omega^3_8 \oplus \Omega^3_{48}$$
$$\Omega^4 = \Omega^4_1 \oplus \Omega^4_7 \oplus \Omega^4_{27} \oplus \Omega^4_{35}$$

where Ω^k_l has (pointwise) dimension l and this decomposition is orthogonal with respect to the metric g. For $k = 2$ and $k = 3$, the explicit descriptions are as follows:

$$\Omega^2_7 = \{\beta \in \Omega^2;\ *(\Phi \wedge \beta) = -3\beta\} \qquad \Omega^2_{21} = \{\beta \in \Omega^2;\ *(\Phi \wedge \beta) = \beta\} \tag{2.2}$$
$$\Omega^3_8 = \{X \lrcorner \Phi;\ X \in \Gamma(TM)\} \qquad \Omega^3_{48} = \{\gamma \in \Omega^3;\ \gamma \wedge \Phi = 0\} \tag{2.3}$$

For $k > 4$, we have $\Omega^k_l = *\Omega^{8-k}_l$.

We need these decompositions in local coordinates. The following proposition is easy to verify.

Proposition 2.1. *Let β_{ij} be a 2-form, γ_{ijk} a 3-form, and X^k a vector field. Then*

$$\beta_{ij} \in \Omega^2_7 \Leftrightarrow \beta_{ab}g^{ap}g^{bq}\Phi_{pqij} = -6\,\beta_{ij} \qquad \beta_{ij} \in \Omega^2_{21} \Leftrightarrow \beta_{ab}g^{ap}g^{bq}\Phi_{pqij} = 2\,\beta_{ij}$$
$$\gamma_{ijk} \in \Omega^3_8 \Leftrightarrow \gamma_{ijk} = X^l\Phi_{ijkl} \qquad \gamma_{ijk} \in \Omega^3_{48} \Leftrightarrow \gamma_{ijk}g^{ia}g^{jb}g^{kc}\Phi_{abcd} = 0$$

and the projection operators π_7 and π_{21} on Ω^2 are given by

$$\pi_7(\beta)_{ij} = \frac{1}{4}\,\beta_{ij} - \frac{1}{8}\,\beta_{ab}g^{ap}g^{bq}\Phi_{pqij} \tag{2.4}$$
$$\pi_{21}(\beta)_{ij} = \frac{3}{4}\,\beta_{ij} + \frac{1}{8}\,\beta_{ab}g^{ap}g^{bq}\Phi_{pqij} \tag{2.5}$$

Remark 2.2. One can show using Proposition 2.1 and Lemma A.1 that if $\beta_{ij} \in \Omega^2_{21}$,

$$\beta_{ab}g^{bl}\Phi_{lpqr} = \beta_{pi}g^{ij}\Phi_{jqra} + \beta_{qi}g^{ij}\Phi_{jrpa} + \beta_{ri}g^{ij}\Phi_{jpqa}$$

which can then be used to show that Ω^2_{21} is a Lie algebra with respect to the commutator of matrices:

$$[\beta, \mu]_{ij} = \beta_{il}g^{lm}\mu_{mj} - \mu_{il}g^{lm}\beta_{mj}$$

In fact, $\Omega^2_{21} \cong \mathfrak{so}(7)$, the Lie algebra of Spin(7).

The decomposition of the space Ω^4 of 4-forms can be understood by considering the infinitesmal action of $\mathrm{GL}(8,\mathbb{R})$ on Φ. Let $A = A^i_l \in \mathfrak{gl}(8,\mathbb{R})$. Hence $e^{At} \in \mathrm{GL}(8,\mathbb{R})$, and we have

$$e^{At}\cdot\Phi = \frac{1}{24}\,\Phi_{ijkl}\,(e^{At}dx^i) \wedge (e^{At}dx^j) \wedge (e^{At}dx^k) \wedge (e^{At}dx^l)$$

Differentiating with respect to t and setting $t = 0$, we obtain:

$$\left.\frac{d}{dt}\right|_{t=0}(e^{At}\cdot\Phi) = A^m_i\,dx^i \wedge \left(\frac{\partial}{\partial x^m} \lrcorner \Phi\right)$$

Now let $A^m_i = g^{mj}A_{ij}$, and decompose $A_{ij} = S_{ij} + C_{ij}$ into symmetric and skew-symmetric parts, where $S_{ij} = \frac{1}{2}(A_{ij} + A_{ji})$ and $C_{ij} = \frac{1}{2}(A_{ij} - A_{ji})$. We have a map

$$\begin{aligned} D &: \mathfrak{gl}(8,\mathbb{R}) \to \Omega^4 \\ D &: A \mapsto \left.\frac{d}{dt}\right|_{t=0}(e^{At}\cdot\Phi) = S_{ij}g^{jm}\,dx^i \wedge \left(\frac{\partial}{\partial x^m} \lrcorner \Phi\right) + C_{ij}g^{jm}\,dx^i \wedge \left(\frac{\partial}{\partial x^m} \lrcorner \Phi\right) \end{aligned}$$

Proposition 2.3. *The kernel of D is isomorphic to the subspace Ω^2_{21}. It is also isomorphic to the Lie algebra $\mathfrak{so}(7)$ of the Lie group* Spin(7) *which is the subgroup of* $\mathrm{GL}(8,\mathbb{R})$ *which preserves Φ.*

Proof. Since we are defining Spin(7) to be the group preserving Φ, the kernel of D is isomorphic to $\mathfrak{so}(7)$ by definition. To show explicitly that this is isomorphic to Ω^2_{21}, suppose that C_{ij} is in Ω^2_{21}. Then $D(C)$ is

$$\frac{1}{24}\left(C^m_i\Phi_{mjkl} + C^m_j\Phi_{imkl} + C^m_k\Phi_{ijml} + C^m_l\Phi_{ijkm}\right)dx^i \wedge dx^j \wedge dx^k \wedge dx^l$$

From Proposition 2.1, we have $C_{ij} = \frac{1}{2} C_{ab} g^{ap} g^{bq} \Phi_{pqij}$. Using this together with the final equation of Lemma A.1, one can compute that

$$C_i^m \Phi_{mjkl} + C_j^m \Phi_{imkl} + C_k^m \Phi_{ijml} + C_l^m \Phi_{ijkm} = \\ -3 \left(C_i^m \Phi_{mjkl} + C_j^m \Phi_{imkl} + C_k^m \Phi_{ijml} + C_l^m \Phi_{ijkm} \right)$$

and hence $D(C) = 0$. Thus Ω^2_{21} is in the kernel of D. We show below that D restricted to Ω^2_7 or to $S^2(T)$ is injective. This completes the proof. $\square$

By counting dimensions, we must have $\Omega^4_7 = D(\Omega^2_7)$ and also $\Omega^4_1 \oplus \Omega^4_{35} = D(S^2)$. We now proceed to establish these explicitly. The proofs of the next two propositions are very similar to analogous results in [8] and are left to the reader.

Proposition 2.4. *Suppose that A_{ij} is a tensor. Consider the 4-form $D(A)$ given by*

$$D(A) = A_{ij} g^{jm}\, dx^i \wedge \left(\frac{\partial}{\partial x^m} \lrcorner \Phi \right)$$

or equivalently

$$D(A)_{ijkl} = A_{im} g^{mn} \Phi_{njkl} + A_{jm} g^{mn} \Phi_{inkl} + A_{km} g^{mn} \Phi_{ijnl} + A_{lm} g^{mn} \Phi_{ijkn} \tag{2.6}$$

Then the Hodge star of $D(A)$ is

$$* D(A) = D(\bar{A}) = \bar{A}_{ij} g^{jm}\, dx^i \wedge \left(\frac{\partial}{\partial x^m} \lrcorner \Phi \right)$$

where $\bar{A}_{ij} = \frac{1}{4} \operatorname{Tr}_g(A) g_{ij} - A_{ji}$. That is, as a matrix, $\bar{A} = \frac{1}{4} \operatorname{Tr}_g(A) I - A^T$.

Proposition 2.5. *Suppose A_{ij} and B_{ij} are two tensors. Let $D(A)$ and $D(B)$ be their corresponding forms in Ω^4. We write $A_{ij} = \frac{1}{8} \operatorname{Tr}_g(A) g_{ij} + (A_0)_{ij} + (A_7)_{ij}$, where A_0 is the symmetric traceless component of A, and A_7 is the component in Ω^2_7. Similarly for B. (We can assume they have no Ω^2_{21} component since that is in the kernel of D.) Then we have*

$$g_\Phi(D(A), D(B)) = \frac{7}{2} \operatorname{Tr}_g(A) \operatorname{Tr}_g(B) + 4 \operatorname{Tr}_g(A_0 B_0) - 16 \operatorname{Tr}_g(A_7 B_7)$$

where $A_0 B_0$ and $A_7 B_7$ mean matrix multiplication.

Corollary 2.6. *The map $D : \mathfrak{gl}(8, \mathbb{R}) \to \Omega^4$ is injective on $S^2 \oplus \Omega^2_7$. It is therefore an isomorphism onto its image, $\Omega^4_1 \oplus \Omega^4_7 \oplus \Omega^4_{35}$.*

Proof. This follows immediately from Proposition 2.5, since if $D(A) = 0$ and A is pure trace, traceless symmetric, or in Ω^2_7, we see that $A = 0$. $\square$

We still need to understand the space Ω^4_{27}. To do this we give another characterization of the space of 4-forms using the Spin(7)-structure, which may be well-known to experts but has apparently not appeared in print before.

Definition 2.7. We define a Spin(7)-equivariant linear operator Λ_Φ on Ω^4 as follows. Let $\sigma \in \Omega^4$. Use the notation $(\sigma \cdot \Phi)_{ijkl}$ to denote $\sigma_{ijmn} g^{mp} g^{nq} \Phi_{pqkl}$. Then $\Lambda_\Phi(\sigma) \in \Omega^4$ is given by

$$(\Lambda_\Phi(\sigma))_{ijkl} = (\sigma \cdot \Phi)_{ijkl} + (\sigma \cdot \Phi)_{iklj} + (\sigma \cdot \Phi)_{iljk} + (\sigma \cdot \Phi)_{jkil} + (\sigma \cdot \Phi)_{jlki} + (\sigma \cdot \Phi)_{klij}$$

We now explain the motivation for introducing this operator Λ_Φ. If $\sigma \in \Omega^4_{27}$, then $g(\sigma, D(A)) = 0$ for all $A \in \mathfrak{gl}(8, \mathbb{R})$ since $D(A) \in \Omega^4_1 \oplus \Omega^4_7 \oplus \Omega^4_{35}$ and the splitting is orthogonal. Writing this in coordinates using (2.6) gives

$$\sigma \in \Omega^4_{27} \qquad \Leftrightarrow \qquad \sigma_{abcd} \Phi_{ijkl} g^{jb} g^{kc} g^{ld} = 0 \quad \text{for all } a, i = 1, \ldots, 8$$

Taking the above expression and contracting it with Φ, and using Lemma A.1, after some laborious calculation one can show that

$$\sigma \in \Omega^4_{27} \qquad \Leftrightarrow \qquad \sigma_{ijkl} = \frac{1}{4} (\Lambda_\Phi(\sigma))_{ijkl}$$

which says that Ω^4_{27} is an eigenspace of Λ_Φ with eigenvalue $+4$. Suppose now that $\sigma = D(A) \in \Omega^4_1 \oplus \Omega^4_7 \oplus \Omega^4_{35}$. Then another brute force calculation using Definition 2.7 and Lemma A.1 shows

$$\Lambda_\Phi(D(A)) = D(6A^T - 6A - 3\,\mathrm{Tr}_g(A) I)$$

and from the above relation it is a simple matter to verify the following characterization of Ω^4.

Proposition 2.8. *The spaces Ω^4_1, Ω^4_7, Ω^4_{27}, and Ω^4_{35} are all eigenspaces of Λ_Φ with distinct eigenvalues. Specifically,*

$$\Omega^4_1 = \{\sigma \in \Omega^4; \Lambda_\Phi(\sigma) = -24\,\sigma\} \qquad \Omega^4_{27} = \{\sigma \in \Omega^4; \Lambda_\Phi(\sigma) = +4\,\sigma\}$$
$$\Omega^4_7 = \{\sigma \in \Omega^4; \Lambda_\Phi(\sigma) = -12\,\sigma\} \qquad \Omega^4_{35} = \{\sigma \in \Omega^4; \Lambda_\Phi(\sigma) = 0\}$$

In addition, we have

$$\Omega^4_1 = \{D(\lambda g); \lambda \in \mathbb{R}\} \qquad \Omega^4_7 = \{D(A_7); A_7 \in \Omega^2_7\} \qquad \Omega^4_{35} = \{D(A_0); A_0 \in S^2_0\}$$

where S^2_0 is the space of symmetric traceless tensors. Also, Proposition 2.4 shows that

$$\Omega^4_+ = \{\sigma \in \Omega^4; *\sigma = \sigma\} = \Omega^4_1 \oplus \Omega^4_7 \oplus \Omega^4_{27} \qquad \Omega^4_- = \{\sigma \in \Omega^4; *\sigma = -\sigma\} = \Omega^4_{35}$$

is the decomposition into self-dual and anti-self dual 4-forms.

Finally, we have the following result, which is also proved using Lemma A.1.

Proposition 2.9. *Let $\sigma \in \Omega^4$. Then if we act on σ by Λ_Φ twice, we have*

$$\begin{aligned} \Lambda_\Phi(\Lambda_\Phi(\sigma))_{ijkl} \quad = \quad & 2\,\Phi_{ijmn} g^{mp} g^{nq} \sigma_{pqrs} g^{ra} g^{sb} \Phi_{abkl} + 2\,\Phi_{ikmn} g^{mp} g^{nq} \sigma_{pqrs} g^{ra} g^{sb} \Phi_{ablj} \\ & + 2\,\Phi_{ilmn} g^{mp} g^{nq} \sigma_{pqrs} g^{ra} g^{sb} \Phi_{abjk} + 24\,\sigma_{ijkl} - 16\,\Lambda_\Phi(\sigma)_{ijkl} \end{aligned}$$

We will need Propositions 2.8 and 2.9 in Section 2.2 to study the torsion of a Spin(7)-structure.

2.2 The torsion tensor of a Spin(7)-structure

In order to define the *torsion tensor* T of a Spin(7)-structure Φ, we need to first study the decomposition of $\nabla_X \Phi$ into its components in Ω^4.

Lemma 2.10. *For any vector field X, the 4-form $\nabla_X \Phi$ lies in the subspace Ω^4_7 of Ω^4. Hence $\nabla\Phi$ lies in the space $\Omega^1_8 \otimes \Omega^4_7$, a 56-dimensional space (pointwise.)*

Proof. Let $X = \frac{\partial}{\partial x^m}$, and consider the 4-form $\nabla_m \Phi$. Then the second equation of Proposition A.2 tells us that $\nabla_m \Phi$ is orthogonal to $S^2 \cong \Omega^4_1 \oplus \Omega^4_{35}$, exactly as in the G_2 case as discussed in [8]. However, we need to work harder to show that there is no Ω^4_{27} component.

The essential reason that $\nabla_X \Phi \in \Omega^4_7$ is because of the way that the 4-form Φ determines the metric g. From [6] (which uses a different orientation convention), we have

$$(u \lrcorner v \lrcorner \Phi) \wedge (w \lrcorner y \lrcorner \Phi) \wedge \Phi = -6\, g(u \wedge v, w \wedge y)\mathsf{vol} + 7\, \Phi(u, v, w, y)\mathsf{vol} \tag{2.7}$$

Taking ∇_X of this identity gives

$$\begin{aligned}(u \lrcorner v \lrcorner \nabla_X \Phi) \wedge (w \lrcorner y \lrcorner \Phi) \wedge \Phi + (u \lrcorner v \lrcorner \Phi) \wedge (w \lrcorner y \lrcorner \nabla_X \Phi) \wedge \Phi& \\ + (u \lrcorner v \lrcorner \Phi) \wedge (w \lrcorner y \lrcorner \Phi) \wedge \nabla_X \Phi &= 7\, \nabla_X \Phi(u, v, w, y)\mathsf{vol}\end{aligned}$$

Now since $*\Phi = \Phi$ and $*(\nabla_X \Phi) = \nabla_X(*\Phi) = \nabla_X \Phi$, this can be written as

$$\begin{aligned}g((u \lrcorner v \lrcorner \nabla_X \Phi) \wedge (w \lrcorner y \lrcorner \Phi), \Phi) + g((u \lrcorner v \lrcorner \Phi) \wedge (w \lrcorner y \lrcorner \nabla_X \Phi), \Phi)& \\ + g(u \lrcorner v \lrcorner \Phi) \wedge (w \lrcorner y \lrcorner \Phi), \nabla_X \Phi) &= 7\, \nabla_X \Phi(u, v, w, y)\end{aligned}$$

We write this expression in coordinates, use Lemma A.1 to simplify the contractions of Φ with itself, and skew-symmetrize the result to obtain

$$\begin{aligned}(\nabla_X \Phi)_{ijkl} \quad = \quad & \frac{1}{12} \left(\Phi_{ijmn} g^{mp} g^{nq} (\nabla_X \Phi)_{pqrs} g^{ra} g^{sb} \Phi_{abkl} + \Phi_{ikmn} g^{mp} g^{nq} (\nabla_X \Phi)_{pqrs} g^{ra} g^{sb} \Phi_{ablj} \right) \\ & + \frac{1}{12} \left(\Phi_{ilmn} g^{mp} g^{nq} (\nabla_X \Phi)_{pqrs} g^{ra} g^{sb} \Phi_{abjk} \right) - \frac{1}{3} \left((\nabla_X \Phi \cdot \Phi)_{ijkl} + (\nabla_X \Phi \cdot \Phi)_{iklj} \right) \\ & - \frac{1}{3} \left((\nabla_X \Phi \cdot \Phi)_{iljk} + (\nabla_X \Phi \cdot \Phi)_{jkil} + (\nabla_X \Phi \cdot \Phi)_{jlki} + (\nabla_X \Phi \cdot \Phi)_{klij} \right)\end{aligned}$$

using the notation of Definition 2.7. In fact the above expression can also be directly verified using the identities of Lemma A.1 and Proposition A.2. Now using Definition 2.7 and Proposition 2.9 this becomes

$$(\nabla_X \Phi)_{ijkl} = -\frac{1}{3} \Lambda_\Phi (\nabla_X \Phi)_{ijkl} + \frac{1}{24} \left(\Lambda_\Phi (\Lambda_\Phi (\nabla_X \Phi))_{ijkl} - 24\, (\nabla_X \Phi)_{ijkl} + 16\, \Lambda_\Phi (\nabla_X \Phi)_{ijkl} \right)$$

Upon simplication, we have finally succeeded in showing that the basic relation (2.7) between the metric and the Spin(7)-structure Φ implies

$$\Lambda_\Phi(\Lambda_\Phi(\nabla_X \Phi)) + 8\, \Lambda_\Phi(\nabla_X \Phi) - 48\, (\nabla_X \Phi) = 0 \qquad \text{for any } X \tag{2.8}$$

Let $\nabla_X \Phi = \sigma_1 + \sigma_7 + \sigma_{27} + \sigma_{35}$ be its decomposition into components, where $\sigma_k \in \Omega^4_k$. Using Proposition 2.8, equation (2.8) says

$$(336)\, \sigma_1 + (0)\, \sigma_7 + (240)\, \sigma_{27} - (48)\, \sigma_{35} = 0$$

which, by linear independence, shows $\sigma_1 = \sigma_{27} = \sigma_{35} = 0$. Therefore $\nabla_X \Phi \in \Omega^4_7$. □

Remark 2.11. The above result was first proved in [2] by Fernàndez, using different methods.

Definition 2.12. Lemma 2.10 says that $\nabla\Phi$ can be written as

$$\nabla_m \Phi_{ijkl} = D(T_m)_{ijkl} = T_{m;ip}g^{pq}\Phi_{qjkl} + T_{m;jp}g^{pq}\Phi_{iqkl} + T_{m;kp}g^{pq}\Phi_{ijql} + T_{m;lp}g^{pq}\Phi_{ijkq}$$

where for each fixed m, $T_{m;ab}$ is in Ω^2_7. This defines the *torsion tensor* T of the Spin(7)-structure, which is an element of $\Omega^1_8 \otimes \Omega^2_7$.

The following lemma gives an explicit formula for $T_{m;ab}$ in terms of $\nabla\Phi$. This will be used in Section 3.1 to derive the evolution equation for the torsion tensor.

Lemma 2.13. *The torsion tensor $T_{m;\alpha\beta}$ is equal to*

$$T_{m;\alpha\beta} = \frac{1}{96}\,(\nabla_m \Phi_{\alpha jkl})\Phi_{\beta bcd}g^{jb}g^{kc}g^{ld} \tag{2.9}$$

Proof. This is a simple computation using Definition 2.12 and the identities in Lemma A.1. □

We close this section with some remarks about the decomposition of T into irreducible components. One can show that $\Omega^1_8 \otimes \Omega^2_7 \cong \Omega^3 = \Omega^3_8 \oplus \Omega^3_{48}$. Therefore the torsion tensor T is actually a 3-form, with two irreducible components. In fact under this isomorphism T is essentially $\delta\Phi$, which is the content of the following result.

Theorem 2.14 (Fernández, 1986)**.** *The* Spin(7)*-structure corresponding to* Φ *is torsion-free if and only if* $d\Phi = 0$*. Since* $*\Phi = \Phi$*, this is equivalent to* $\delta\Phi = 0$*.*

Suppose M is simply-connected, as it must be to admit a metric with holonomy exactly equal to Spin(7) in the compact case (see [5].) Then as in the G_2 case, which is described in [8], the component of the torsion in Ω^3_8 can always be conformally scaled away, once we have made the Ω^3_{48} component vanish, without changing that other component. Therefore in principle we can restrict our attention to trying to make the Ω^3_{48} component of the torsion vanish, although it is not clear if this is really a simplification. We will not pursue this here.

3 General flows of Spin(7)-structures

In this section we derive the evolution equations for a general flow $\frac{\partial}{\partial t}\Phi$ of a Spin(7)-structure Φ. Let $A_{ij} = h_{ij} + X_{ij}$, where $h_{ij} \in S^2$ and $X_{ij} \in \Omega^2_7$. Then from the discussion in Section 2.1, a general variation of Φ can be written as $\frac{\partial}{\partial t}\Phi = D(A)$. In coordinates, using (2.6), this is

$$\boxed{\frac{\partial}{\partial t}\Phi_{ijkl} \;=\; A_{im}g^{mn}\Phi_{njkl} + A_{jm}g^{mn}\Phi_{inkl} + A_{km}g^{mn}\Phi_{ijnl} + A_{lm}g^{mn}\Phi_{ijkn}} \tag{3.1}$$

The first thing we need to do is to derive the evolution equations for the metric g and objects related to the metric, specifically the volume form vol and the Christoffel symbols Γ^k_{ij}. We do this using a much simpler argument than that presented in [8] for the G_2 case. This method works for that case as well.

Proposition 3.1. *The evolution of the metric g_{ij} under the flow* (3.1) *is given by*

$$\boxed{\frac{\partial}{\partial t}g_{ij} = 2\,h_{ij}} \tag{3.2}$$

Proof. We want to know what the first order variation of the metric g_Φ is, given a first order variation $D(A)$ of the Spin(7)-structure Φ. It suffices to consider any path $\Phi(t)$ of Spin(7)-structures that satisfies $\frac{\partial}{\partial t}\big|_{t=0}\Phi(t) = D(A)$. We take

$$\Phi(t) = e^{At}\cdot\Phi = \frac{1}{24}\,\Phi_{ijkl}\,(e^{At}dx^i)\wedge(e^{At}dx^j)\wedge(e^{At}dx^k)\wedge(e^{At}dx^l)$$

Then if $g = g_{ij}dx^i dx^j$ is the metric of $\Phi = \Phi(0)$, it is easy to see that the metric $g(t)$ of $\Phi(t)$ is

$$g(t) = g_{ij}(e^{At}dx^i)(e^{At}dx^j)$$

Now we differentiate

$$\begin{aligned}\frac{\partial}{\partial t}\bigg|_{t=0} g(t) &= g_{ij}\,(A^i_k dx^k)dx^j + g_{ij}\,dx^i(A^j_l dx^l) = A_{kj}\,dx^k dx^j + A_{li}\,dx^i dx^l \\ &= (A_{ij}+A_{ji})\,dx^i dx^j = 2\,h_{ij}\,dx^i dx^j\end{aligned}$$

since h is the symmetric part of A. This completes the proof. □

Corollary 3.2. *The evolution of the inverse g^{ij} of the metric, the volume form* vol*, and the Christoffel symbols Γ^k_{ij}, under the flow* (3.1)*, are given by*

$$\frac{\partial}{\partial t}g^{ij} = -2\,h^{ij} \qquad \frac{\partial}{\partial t}\mathsf{vol} = \mathrm{Tr}_g(h)\,\mathsf{vol} \qquad \frac{\partial}{\partial t}\Gamma^k_{ij} = g^{kl}\,(\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij})$$

Proof. This is a standard result. □

3.1 Evolution of the torsion tensor

In this section we derive the evolution equation for the torsion tensor T of Φ under the general flow (3.1). We begin with the evolution of $\nabla_m\Phi_{ijkl}$.

Lemma 3.3. *The evolution of $\nabla_m\Phi_{ijkl}$ under the flow* (3.1) *is given by*

$$\begin{aligned}\frac{\partial}{\partial t}(\nabla_m\Phi_{ijkl}) = {} & A_{ip}g^{pq}(\nabla_m\Phi_{qjkl}) + A_{jp}g^{pq}(\nabla_m\Phi_{iqkl}) + A_{kp}g^{pq}(\nabla_m\Phi_{ijql}) + A_{lp}g^{pq}(\nabla_m\Phi_{ijkq}) \\ & + (\nabla_p h_{im})g^{pq}\Phi_{qjkl} + (\nabla_p h_{jm})g^{pq}\Phi_{iqkl} + (\nabla_p h_{km})g^{pq}\Phi_{ijql} + (\nabla_p h_{lm})g^{pq}\Phi_{ijkq} \\ & - (\nabla_i h_{pm})g^{pq}\Phi_{qjkl} - (\nabla_j h_{pm})g^{pq}\Phi_{iqkl} - (\nabla_k h_{pm})g^{pq}\Phi_{ijql} - (\nabla_l h_{pm})g^{pq}\Phi_{ijkq} \\ & + (\nabla_m X_{ip})g^{pq}\Phi_{qjkl} + (\nabla_m X_{jp})g^{pq}\Phi_{iqkl} + (\nabla_m X_{kp})g^{pq}\Phi_{ijql} + (\nabla_m X_{lp})g^{pq}\Phi_{ijkq}\end{aligned}$$

where $A_{ij} = h_{ij} + X_{ij} \in S^2 \oplus \Omega^2_7$.

Proof. Recall that

$$\nabla_m\Phi_{ijkl} = \frac{\partial}{\partial x^m}\Phi_{ijkl} - \Gamma^n_{mi}\Phi_{njkl} - \Gamma^n_{mj}\Phi_{inkl} - \Gamma^n_{mk}\Phi_{ijnl} - \Gamma^n_{ml}\Phi_{ijkn}$$

so if we differentiate this equation with respect to t and simplify, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}(\nabla_m\Phi_{ijkl}) = {} & \nabla_m\left(\frac{\partial}{\partial t}\Phi_{ijkl}\right) - \left(\frac{\partial}{\partial t}\Gamma^n_{mi}\right)\Phi_{njkl} \\ & - \left(\frac{\partial}{\partial t}\Gamma^n_{mj}\right)\Phi_{inkl} - \left(\frac{\partial}{\partial t}\Gamma^n_{mk}\right)\Phi_{ijnl} - \left(\frac{\partial}{\partial t}\Gamma^n_{ml}\right)\Phi_{ijkn}\end{aligned}$$

Now we substitute (3.1) and use Corollary 3.2. After we use the product rule on the first term, all the terms involving $\nabla_m h$ cancel in pairs. The result now follows. $\square$

Theorem 3.4. *The evolution of the torsion tensor $T_{m;\alpha\beta}$ under the flow* (3.1) *is given by*

$$\boxed{\frac{\partial}{\partial t} T_{m;\alpha\beta} = A_{\alpha p} g^{pq} T_{m;q\beta} - A_{\beta p} g^{pq} T_{m;q\alpha} + \pi_7(\nabla_\beta h_{\alpha m} - \nabla_\alpha h_{\beta m} + \nabla_m X_{\alpha\beta})} \tag{3.3}$$

where $A_{ij} = h_{ij} + X_{ij}$ is the element of $S^2 \oplus \Omega^2_7$ corresponding to the flow of Φ, and π_7 denotes the projection onto Ω^2_7 of the tensor skew-symmetric in α, β for fixed m.

Proof. This is a long computation, but is similar in spirit to the analogous result for G_2-structures in [8]. We will describe the main steps, and leave the details to the reader. Begin with Lemma 2.13, and differentiate to obtain:

$$\begin{aligned} \frac{\partial}{\partial t} T_{m;\alpha\beta} \quad &= \quad \frac{1}{96} \left(\frac{\partial}{\partial t} \nabla_m \Phi_{\alpha jkl} \right) \Phi_{\beta bcd} g^{jb} g^{kc} g^{ld} + \frac{1}{96} (\nabla_m \Phi_{\alpha jkl}) \left(\frac{\partial}{\partial t} \Phi_{\beta bcd} \right) g^{jb} g^{kc} g^{ld} \\ &\quad - \frac{6}{96} (\nabla_m \Phi_{\alpha jkl}) \Phi_{\beta bcd} h^{jb} g^{kc} g^{ld} \end{aligned} \tag{3.4}$$

where we have used $\frac{\partial}{\partial t} g^{ij} = -2\, h^{ij}$ from Corollary 3.2. Recall that for a tensor B_{ij} we defined

$$D(B)_{ijkl} = B_{ip} g^{pq} \Phi_{qjkl} + B_{jp} g^{pq} \Phi_{iqkl} + B_{kp} g^{pq} \Phi_{ijql} + B_{lp} g^{pq} \Phi_{ijkq}$$

Let us define a similar shorthand notation $D_m(B)$ to denote

$$D_m(B)_{ijkl} = B_{ip} g^{pq} \nabla_m \Phi_{qjkl} + B_{jp} g^{pq} \nabla_m \Phi_{iqkl} + B_{kp} g^{pq} \nabla_m \Phi_{ijql} + B_{lp} g^{pq} \nabla_m \Phi_{ijkq}$$

Then Lemma 3.3 says that

$$\frac{\partial}{\partial t} \nabla_m \Phi_{ijkl} = D_m(A) + D(B) \tag{3.5}$$

where we define

$$B_{\alpha\beta} = \nabla_\beta h_{\alpha m} - \nabla_\alpha h_{\beta m} + \nabla_m X_{\alpha\beta} \tag{3.6}$$

We also have $\frac{\partial}{\partial t}\Phi = D(A)$. Therefore (3.4) becomes

$$\begin{aligned} \frac{\partial}{\partial t} T_{m;\alpha\beta} \quad &= \quad \frac{1}{96} D_m(A)_{\alpha jkl} \Phi_{\beta bcd} g^{jb} g^{kc} g^{ld} + \frac{1}{96} D(B)_{\alpha jkl} \Phi_{\beta bcd} g^{jb} g^{kc} g^{ld} \\ &\quad + \frac{1}{96} (\nabla_m \Phi_{\alpha jkl}) D(A)_{\beta bcd} g^{jb} g^{kc} g^{ld} - \frac{6}{96} (\nabla_m \Phi_{\alpha jkl}) \Phi_{\beta bcd} h^{jb} g^{kc} g^{ld} \end{aligned} \tag{3.7}$$

We will break up the computation into several manageable pieces. First, we need the following identity. If $A = h + X \in S^2 \oplus \Omega^2_7$, then:

$$\begin{aligned} D(A)_{ijkl} \Phi_{abcd} g^{kc} g^{ld} \quad &= \quad 4\, h_{ia} g_{jb} - 4\, h_{ib} g_{ja} + 4\, h_{jb} g_{ia} - 4\, h_{ja} g_{ib} + 2\, \mathrm{Tr}_g(h)(g_{ia} g_{jb} - g_{ib} g_{ja} - \Phi_{ijab}) \\ &\quad + 16\, X_{ia} g_{jb} - 16\, X_{ib} g_{ja} + 16\, X_{jb} g_{ia} - 16\, X_{ja} g_{ib} \\ &\quad - 2\, A_{ip} g^{pq} \Phi_{qjab} + 2\, A_{jp} g^{pq} \Phi_{qiab} + 2\, A_{pa} g^{pq} \Phi_{qbij} - 2\, A_{pb} g^{pq} \Phi_{qaij} \end{aligned} \tag{3.8}$$

which can be proved using Lemma A.1. Also, it is easy to check that if X and Y are both in Ω^2_7, then

$$X_{ip} g^{pq} Y_{qj} g^{ia} g^{jb} \Phi_{abkl} = X_{kp} g^{pq} Y_{ql} - Y_{kp} g^{pq} X_{ql} \tag{3.9}$$

which essentially says that the Lie bracket of two elements of Ω^2_7 is always in Ω^2_{21}. Now using the identities (3.8) and (3.9), and Lemma A.1 again, along with some patience, one can establish the following four expressions:

$$\begin{aligned}
D(A)_{\alpha jkl} \Phi_{\beta bcd} g^{jb} g^{kc} g^{ld} &= 24\, h_{\alpha\beta} + 18\, \mathrm{Tr}_g(h)\, g_{\alpha\beta} + 96\, X_{\alpha\beta} \\
D_m(A)_{\alpha jkl} \Phi_{\beta bcd} g^{jb} g^{kc} g^{ld} &= 48\, (h_{\alpha p} g^{pq} T_{m;q\beta} + h_{\beta p} g^{pq} T_{m;q\alpha} + \mathrm{Tr}_g(h)\, T_{m;\alpha\beta} - g(X, T_m) g_{\alpha\beta}) \\
(\nabla_m \Phi_{\alpha jkl}) D(A)_{\beta bcd} g^{jb} g^{kc} g^{ld} &= 48\, (-h_{\alpha p} g^{pq} T_{m;q\beta} - h_{\beta p} g^{pq} T_{m;q\alpha} + \mathrm{Tr}_g(h)\, T_{m;\alpha\beta} + g(X, T_m) g_{\alpha\beta}) \\
&\quad + 96\, (X_{\alpha p} g^{pq} T_{m;q\beta} - X_{\beta p} g^{pq} T_{m;q\alpha}) \\
(\nabla_m \Phi_{\alpha jkl}) \Phi_{\beta bcd} h^{jb} g^{kc} g^{ld} &= 16\, (-h_{\alpha p} g^{pq} T_{m;q\beta} - h_{\beta p} g^{pq} T_{m;q\alpha} + \mathrm{Tr}_g(h)\, T_{m;\alpha\beta})
\end{aligned}$$

Now we use the above four expressions to simplify equation (3.7). We need to substitute B as defined in (3.6) for A when we use the first of these expressions. After much cancellation and collecting like terms, we are left with exactly (3.3). □

We remark that, just as in the G_2 case, the terms with ∇h and with ∇X play quite different roles in the evolution of the torsion tensor in equation (3.3). One hopes that it is possible to choose h and X in terms of T and possibly also ∇T so that the evolution equations have nice properties. In particular we would like the equation to be parabolic transverse to the action of the diffeomorphism group, for short-time existence. Ideally such a flow exists where the L^2-norm $||T||$ of the torsion decreases. These are questions for future research.

4 Bianchi-type identity and curvature formulas

In this section, we apply the evolution equation (3.3) to derive a Bianchi-type identity for manifolds with Spin(7)-structure. This yields explicit formulas for the Ricci tensor and part of the Riemann curvature tensor in terms of the torsion tensor. As the calculations here are extremely similar to those in [8], we will be brief.

Proposition 4.1. *The diffeomorphism invariance of the metric g as a function of the 4-form Φ is equivalent to the vanishing of the $\Omega^4_1 \oplus \Omega^4_{35}$ component of $\nabla_Y \Phi$ for any vector field Y. This is the fact which was proved earlier in Lemma 2.10.*

Proof. The proof is identical to the G_2 case. In both cases it is due to the fact that the evolution of the metric g depends only on the symmetric part h of $A = h + X$. Notice that in the Spin(7) case, there is a stronger result that the Ω^4_{27} component of $\nabla_Y \Phi$ also vanishes, Lemma 2.10, which does not follow from here. □

Theorem 4.2. *The diffeomorphism invariance of the torsion tensor T as a function of the 4-form Φ is equivalent to the following identity:*

$$\boxed{\nabla_q T_{p;\alpha\beta} - \nabla_p T_{q;\alpha\beta} = \frac{1}{4} R_{pq\alpha\beta} - \frac{1}{8} R_{pqij} g^{ia} g^{jb} \Phi_{ij\alpha\beta} + 2\, T_{q;\alpha m} g^{mn} T_{p;n\beta} - 2\, T_{p;\alpha m} g^{mn} T_{q;n\beta}} \tag{4.1}$$

Proof. The proof is very similar to the analogous result for G_2-structures described in [8], and is left to the reader. The identity (4.1) can also be established directly by using (2.9), Lemma A.1, and the Ricci identities. □

We now examine some consequences of Theorem 4.2. For i and j fixed, the Riemann curvature tensor R_{ijkl} is skew-symmetric in k and l. Hence we can use the decomposition of Ω^2 to write it as

$$R_{ijkl} = (\pi_7(\mathrm{Riem}))_{ijkl} + (\pi_{21}(\mathrm{Riem}))_{ijkl}$$

where by equation (2.4), we have

$$(\pi_7(\mathrm{Riem}))_{ijkl} = \frac{1}{4} R_{ijkl} - \frac{1}{8} R_{ijab} g^{ap} g^{bq} \Phi_{pqkl} \tag{4.2}$$

Therefore the identity (4.1) says that

$$(\pi_7(\mathrm{Riem}))_{pq\alpha\beta} = \nabla_q T_{p;\alpha\beta} - \nabla_p T_{q;\alpha\beta} + 2\left(T_{p;\alpha m} g^{mn} T_{q;n\beta} - T_{q;\alpha m} g^{mn} T_{p;n\beta}\right) \tag{4.3}$$

Corollary 4.3. *If Φ is torsion-free, then the Riemann curvature tensor $R_{ijkl} \in S^2(\Omega^2)$ actually takes values in $S^2(\Omega^2_{21})$, where $\Omega^2_{21} \cong \mathfrak{so}(7)$, the Lie algebra of* Spin(7).

Proof. Setting $T = 0$ in (4.3) shows the for fixed i, j, we have $R_{ijkl} \in \Omega^2_{21}$ as a skew-symmetric tensor in k, l. The result now follows from the symmetry $R_{ijkl} = R_{klij}$. □

Remark 4.4. This result is well-known. When $T = 0$, the Riemannian holonomy of the metric g_Φ is contained in the group Spin(7). By the Ambrose-Singer holonomy theorem, the Riemann curvature tensor of the metric is thus an element of $S^2(\mathfrak{so}(7))$.

Lemma 4.5. *Let $Q_{ijkl} = R_{ijab} g^{ap} g^{bq} \Phi_{pqkl}$. Then we have $Q_{ijkl} g^{il} = 0$.*

Proof. This is identical to the G_2 case proved in [8]. □

From Lemma 4.5 and equation (4.2), we see that the Ricci tensor R_{jk} can be expressed as

$$R_{jk} = R_{ijkl} g^{il} = 4\,(\pi_7(\mathrm{Riem}))_{ijkl} g^{il} \tag{4.4}$$

Proposition 4.6. *Given a* Spin(7)*-structure Φ with torsion tensor $T_{m;\alpha\beta}$, its associated metric g has Ricci curvature R_{jk} given by*

$$R_{jk} = 4\, g^{il} \nabla_i T_{j;lk} - 4\, \nabla_j \left(g^{il} T_{i;lk}\right) + 8\, T_{j;mk} T_{i;nl} g^{mn} g^{il} - 8\, T_{j;ml} T_{i;nk} g^{mn} g^{il}$$

Proof. This follows immediately from equations (4.3) and (4.4). □

Corollary 4.7. *The metric of a torsion-free* Spin(7)*-structure is necessarily Ricci-flat. This is classical, originally proved by Bonan. Here we see a direct proof of this fact.*

Remark 4.8. Some formulas relating the Ricci tensor and the torsion on a manifold with Spin(7)-structure have also been obtained by Ivanov in [3].

A Identites in Spin(7)-geometry

In this appendix we collect several identities involving the 4-form Φ of a Spin(7)-structure. They are derived by methods analogous to those for the G_2 case as explained in [8], so we omit the proofs.

In local coordinates $x^1, x^2, \ldots, x^8$, the 4-form Φ is

$$\Phi = \frac{1}{24}\Phi_{ijkl}\, dx^i \wedge dx^j \wedge dx^k \wedge dx^l$$

where Φ_{ijkl} is totally skew-symmetric. The metric is given by $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$.

Lemma A.1. *The following identities hold:*

$$\begin{aligned}
\Phi_{ijkl}\Phi_{abcd}g^{ia}g^{jb}g^{kc}g^{ld} &= 336 \\
\Phi_{ijkl}\Phi_{abcd}g^{jb}g^{kc}g^{ld} &= 42g_{ia} \\
\Phi_{ijkl}\Phi_{abcd}g^{kc}g^{ld} &= 6g_{ia}g_{jb} - 6g_{ib}g_{ja} - 4\Phi_{ijab} \\
\Phi_{ijkl}\Phi_{abcd}g^{ld} &= g_{ia}g_{jb}g_{kc} + g_{ib}g_{jc}g_{ka} + g_{ic}g_{ja}g_{kb} \\
&\quad - g_{ia}g_{jc}g_{kb} - g_{ib}g_{ja}g_{kc} - g_{ic}g_{jb}g_{ka} \\
&\quad - g_{ia}\Phi_{jkbc} - g_{ja}\Phi_{kibc} - g_{ka}\Phi_{ijbc} \\
&\quad - g_{ib}\Phi_{jkca} - g_{jb}\Phi_{kica} - g_{kb}\Phi_{ijca} \\
&\quad - g_{ic}\Phi_{jkab} - g_{jc}\Phi_{kiab} - g_{kc}\Phi_{ijab}
\end{aligned}$$

Proposition A.2. *The following identities hold:*

$$\begin{aligned}
(\nabla_m \Phi_{ijkl})\Phi_{abcd}g^{ia}g^{jb}g^{kc}g^{ld} &= 0 \\
(\nabla_m \Phi_{ijkl})\Phi_{abcd}g^{jb}g^{kc}g^{ld} &= -\Phi_{ijkl}(\nabla_m \Phi_{abcd})g^{jb}g^{kc}g^{ld} \\
(\nabla_m \Phi_{ijkl})\Phi_{abcd}g^{kc}g^{ld} &= -\Phi_{ijkl}(\nabla_m \Phi_{abcd})g^{kc}g^{ld} - 4\,\nabla_m \Phi_{ijab}
\end{aligned}$$

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